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The construction of Lyapunov functions with sign-definite derivative for systems satisfying the Barbashin–Krasovskii theorem☆

A.M. Kovalev

Donetsk, Ukraine

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ABSTRACT

The asymptotic stability of the zero solution of autonomous systems of differential equations is considered. For systems satisfying the Barbashin–Krasovskii theorem positive-definite functions are constructed having a negative-definite derivative. The investigation is based on the method of invariant relations.

The method of Lyapunov functions is one of the most effective methods for the qualitative investigation of complex dynamic and controlled systems. As Barbashin showed,[1] the importance of this method goes far beyond the mere possibility of establishing the fact of the stability or instability of a system. A successfully constructed Lyapunov functions for a specific control system enables one to solve a whole range of problems: to estimate the change in the regulated quantity, to estimate the time for which a transient occurs, the optimization of the integral quadratic error, etc. Using Lyapunov functions one can estimate the region of attraction, solve the problem of stability “in the large”, the problem of the existence or absence of periodic solutions, etc.

Unsurprisingly, the problems of constructing Lyapunov functions remain urgent at the present time. One such problem is the construction of a Lyapunov function which ensures the asymptotic stability for systems satisfying the Barbashin–Krasovskii theorem. In the first paper,[2] which announced this result, it was noted that if the set M, which occurs in the formulation of the theorem is specified by the equation $\psi(x_1, x_2,\ldots, x_n) = 0$, the non-vanishing of the expression $\sum_{i=1}^{n}(\partial\psi/\partial x_i)X_i$ everywhere in M, apart from the origin of coordinates, ensures that there is no half-trajectory lying in M (the system $\dot{x}_i = X_i(x_1, \ldots, x_n)(i = 1, \ldots, n)$ was considered). But the assertion in later publications (see, for example, Ref. 1) that this condition is easily verified, is not so obvious. An exact solution of this problem can be obtained using the method of invariant relations, developed in a number of papers (Refs. 3-5). The solution is constructed using higher derivatives of Lyapunov functions. This confirms the correctness of the observation[6] that the possibilities of obtaining new criteria of stability with several Lyapunov functions and using their higher-order derivatives are obviously far from exhausted. We should add that this observation is also timely today. One of the recent realizations of this idea can be found in Ref. 7.

Below, using the method of invariant relations, we propose a procedure for checking that the conditions of the Barbashin–Krasovskii theorm are satisfied. A form of the equations is proposed for a system which satisfies the Barbashin–Krasovskii theorem, using which one can obtain an explicit expression for Lyapunov function with a sign-definite derivative. Two illustrative examples are considered.

1. The Barbashin–Krasovskii theorem and the method of invariant relations

Consider the following system of differential equations

$$\dot{x} = f(x), \quad f(0) = 0; \quad x \in D \subset R^n, \quad t \in [t_0, \infty) \tag{1.1}$$

where D is a certain neighbourhood of zero, and the function $f(x)$ is assumed to be continuously differentiable a sufficient number of times for $x \in D$. The dot denotes differentiation with respect to time t of the dependent variable x, and also the functions $V(x)$ by virtue of system (1.1): $\dot{V}(x) = (\nabla V(x), f(x))$. Here ∇ is the differentiation operator, and as it applies to a scalar function it gives the gradient, while as applied to a vector function it gives the Jacobi matrix, and the symbol (,) denotes the scalar product.

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E-mail address: kovalev@iamm.ac.donetsk.ua.

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Asymptotic stability of the zero solution can be established using the Barbashin–Krasovskii theorem [Refs. 1,2].

Theorem 1. If a positive function $V(x)$ definitely exists, such that $\dot{V}(x)$ is a negative constant function and the set $M = \{x : \dot{V}(x) = 0\}$ does not contain integer half-trajectories, apart from the point $x = 0$, the zero solution of system (1.1) is asymptotically stable.

We will assume that the Lyapunov function in Theorem 1 is continuously differentiable a sufficient number of times. Then the equation $\dot{V}(x) = 0$ defines, in the neighbourhood of zero, a certain set of surfaces M_i, described by the equations $\varphi_i(x) = 0 (i = 1,\ldots, s)$, where $\varphi_i(x)$ are k_i-dimensional differentiable vector functions, and $\nabla\varphi_i(x) \neq 0$ for $x \in M$, apart from $x = 0$.

We will check using the method of invariant relations, whether the set $M = \bigcup_{i=1}^{s} M_i$ contains integer half-trajectories, i.e., a certain invariant manifold. We will present two theorems, which are necessary to solve this problem.

Theorem 2 ([5]). The invariant manifold of system (1.1), generated by the invariant relation $\varphi(x) = 0$, is defined by the equations

$$\varphi^{(i)}(x) = 0 \ (i = 0, 1, \ldots, l-1) \tag{1.2}$$

where l is the number of functionally independent functions in the sequence $\varphi(x), \dot{\varphi}(x), \ddot{\varphi}(x), \ldots$.

Theorem 3 ([4]). In order that the equations $V_i(x) = 0$ $(i = 1,\ldots, l)$ should define the invariant manifold of system (1.1), it is necessary and sufficient that the functions $V_i(x)$ should satisfy the following system of partial differential equations

$$(f(x), \nabla V_i(x)) = \sum_{j=1}^{l} \lambda_{ij}(x) V_j(x), \quad i = 1, \ldots, l$$

where the functions $\lambda_{ij}(x)$ have no singularities in the region considered.

Theorem 2 enables us to establish whether the set $M = \{x: \varphi(x) = 0\}$ contains the invariant manifold of system (1.1). To do this we must consider the following system of equations

$$\varphi(x) = 0, \quad \dot{\varphi}(x) = 0, \ldots, \varphi^{(n-1)}(x) = 0$$

If this system has no other solutions, apart from the zero solution $x = 0$, system (1.1) has no invariant manifolds belonging to the set M, apart from $x = 0$. Here it should be noted that it is important that the condition $\nabla\varphi(x) \neq 0$ must be satisfied for $x \in M$, since its violation may lead to compatibility of the system considered, but no invariant manifold.

In fact, suppose, for example, that $\varphi(x) = \psi^2(x)$ $(\varphi \in R^1)$. It is obvious that system (1.2) for $l = 1$ is compatible for any function $\psi(x)$, but the manifold $\psi(x) = 0$ is not an invariant manifold of system (1.1) if the function $\psi(x)$ is arbitrary. The reason for this is the fact that $\nabla\varphi(x) = 0$ in the set $M = \{x: \varphi(x) = 0\}$. This must be taken into account when using the derivative of the Lyapunov function to check the conditions of the Barbashin–Krasovskii theorem. This check must be made in two stages. At the first stage it is necessary to obtaining the equations of the surfaces M_i, on which $\dot{V}(x) = 0 : \varphi_i(x) = 0 (i = 1, \ldots, s), \nabla\varphi_i(x) \neq 0$ for $x \in M$.

At the second stage one must establish that the system of equations

$$\varphi_i(x) = 0, \quad \dot{\varphi}_i(x) = 0, \ldots, \quad \varphi_i^{(n-1)}(x) = 0; \quad i = 1, \ldots, s \tag{1.3}$$

has no solutions differing from $x = 0$. Taking this into account, we can formulate Theorem 1 as follows.

Theorem 4. Suppose a positive definite function $V(x)$ exists such that $\dot{V}(x)$ is a negative constant function, while the set $M = \{x : \dot{V}(x) = 0\}$ is the union $M = \bigcup_{i=1}^{s} M_i$ of the surfaces M_i, specified by the equations $\varphi_i(x) = 0$ $(i = 1,\ldots, s)$, where φ_i are n_i-dimensional vector functions, where the systems of equations (1.3) have no solutions, differing from $x = 0$. Then, the zero solution of system (1.1) is asymptotically stable.

We will apply Theorem 4 to the two examples considered.

Example 1 *([7])*. Consider the system

$$\dot{x} = -x^3 + ax^2y + bxy^2 + cy^3, \quad \dot{y} = -ax^3 - bx^2y - cxy^2 \tag{1.4}$$

We will take $V = x^2 + y^2$ as the Lyapunov function. We obtain $\dot{V} = -2x^4$. The set M is defined by the equation $x = 0$. The system of equations $x = 0, \dot{x} = -x^3 + ax^2y + bxy^2 + cy^3 = 0$ has only a zero solution. By Theorem 4 the zero solution of system (1.4) is asymptotically stable. Note that, in this example, exactly this case occurs when it is necessary to transfer from the function $\dot{V} = -2x^4$ to the function $\varphi = x$, for which $\nabla\varphi \neq 0$ in M. (The author of this example, A. A. Tsygankov, used $V^{(4)}(x)$ in order to eliminate this singularity.)

Example 2. Consider the system

$$\begin{aligned} &\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 - a x_2^2 x_4 + b x_3^2 \\ &\dot{x}_3 = \omega x_4 + c x_4^2 - b x_2 x_3, \quad \dot{x}_4 = -\omega x_3 - c x_3 x_4 + a x_2^3 + \lambda_4 x_4^3 \end{aligned} \tag{1.5}$$

We will take $V = x_1^2 + x_2^2 + x_3^2 + x_4^2$ as the Lyapunov function. For the derivative $\dot{V} = 2\lambda_1 x_1^2 + 2\lambda_2 x_2^2 + 2\lambda_4 x_4^4$ we obtain $\dot{V} \leq 0$ when $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_4 < 0$. The set M is defined by the equations

$$\varphi_1 = x_1 = 0, \quad \varphi_2 = x_2 = 0, \quad \varphi_3 = x_4 = 0$$

supplementing which with one more equation

$$\dot{\varphi}_2 = \dot{x}_2 = \lambda_2 x_2 - a x_2^2 x_4 + b x_3^2 = 0$$

we obtain that when $b \neq 0$ the system of equations

$$\varphi = 0, \quad \dot{\varphi} = 0, \quad \ddot{\varphi} = 0, \quad \varphi^{(3)} = 0$$

has only the zero solution (here the vector functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ is denoted by φ). We conclude from Theorem 4 that the zero solution of system (1.1) is asymptotically stable.

2. Auxiliary functions

When there is a Lyapunov function which satisfies the Barbashin–Krasovskii theorem, the Lyapunov function with sign-definite derivative can be constructed by adding to this function, which we use as the initial V_s, an additional function V_a, the derivative of which $\dot{V}_a$ is positive in the set $M = \{x : \dot{V}_s = 0\}$. We must choose the function V_a to be fairly small so that it cannot affect the sign-definiteness and sign-constancy of the functions V_s, $\dot{V}_s$. It turns out that the conditions imposed by the Barbashin–Krasovskii theorem on system (1.1) enables this to be done.

We will consider the case when the set $M = \{x : \dot{V} = 0\}$ is specified by a single vector function $\varphi(x) = 0$, and the presence in it of an invariant manifold is established by the first two terms of sequence (1.2).

Lemma 1. Suppose system (1.1) satisfies the Barbashin–Krasovskii theorem and the set $M = \{x : \dot{V}(x) = 0\}$ is defined by one vector function: $\varphi(x) = 0$, for which the assumption made above holds.

Then the derivative of the function

$$V_a = (\nabla\varphi(x), f(x))^{2m}((\nabla\varphi(x), f(x)), \varphi(x)) \tag{2.1}$$

takes a positive value $\dot{V}_a = (\nabla\varphi(x), f(x))^{2m+2}$ in the set M.

Proof. We will use the fact that $\dot{\varphi} = (\nabla\varphi, f)$ and we have $\varphi = 0$ in the set M. Then, we obtain by direct differentiation $\dot{\varphi} = (\nabla\varphi(x), f(x))^{2m+2}$ for $x \in M$. The inequality $(\nabla\varphi f) \neq 0$ when $\varphi(x) = 0$ follows from the fact that the assumption $(\nabla\varphi f) = 0$ when $\varphi(x) = 0$ leads to the fact that the system $\varphi = 0$, $\dot{\varphi} = (\nabla\varphi, f) = 0$ is compatible and allows of a non-zero solution, which, by Theorem 2, denotes the existence in the set M of an invariant manifold, and this contradicts the satisfaction of the conditions of the Barbashin–Krasovskii theorem.

When the conditions of the Barbashin–Krasovskii theorem are satisfied, system (1.1) can be converted to a form which is more convenient for constructing the Lyapunov function with sign-definite derivative.

Lemma 2. Suppose a positive-definite function $V(x)$ with negative constant derivative exists for system (1.1), which vanishes in the set M, and which does not contain integer half-trajectories. Then system (1.1) can be converted to the form

$$\dot{x} = f_M(x) + f_N(x) \tag{2.2}$$

where the function $f_M(x)$ vanishes in the set M, while the function $f_N(x)$ is non-zero in the set M.

The existence of such a representation follows from the fact that if the function $f_N(x)$ did not exist, i.e. $\dot{x} = f_M(x)$, the derivatives of all the functions $\varphi_i(x)$, which define the set M, would vanish in the set M by virtue of the fact that $\dot{\varphi}_i(x) = (\nabla\varphi_i(x), f_M(x))$. By virtue of Theorem 1 this would mean that the set M contains an invariant manifold, which contradicts the above assumption.Representation (2.2) enables us to simplify function (2.1).

Lemma 3. Suppose system (1.1) satisfies the Barbashin–Krasovskii theorem and is reduced to the form (2.2), and the set $M = \{x : \dot{V}(x) = 0\}$ is defined by a single vector function: $\varphi(x) = 0$. Then the derivative of the function

$$V_a = (\nabla\varphi(x), f_N(x))^{2m}((\nabla\varphi(x), f_N(x)), \varphi(x)) \tag{2.3}$$

takes the positive value $\dot{V}_a = (\nabla\varphi(x), f_N(x))^{2m+2}$ in the set M.

We will use Lemmas 1–3 to investigate the systems of Examples 1 and 2.

Example 1. For system (1.4), taking into account the equality $\varphi = x$, we obtain from formula (2.1)

$$V_a = (-x^3 + ax^2y + bxy^2 + cy^3)^{2m+1}x$$

Representation (2.2) for system (1.4) has the form

$$f_M = (-x^3 + ax^2y + bxy^2, -ax^3 - bx^2y - cxy^2)^{\mathrm{T}}, \quad f_N = (cy^3, 0)^{\mathrm{T}}$$

while expression (2.3) for the function V_a takes the form $V_a = (cy^3)^{2m+1}x$. The derivatives of the functions V_a in the set $M = \{(x,y) : x = 0\}$ take the value $\dot{V}_a = (cy^3)^{2m+2}$.

Example 2. For system (1.5) we have $\varphi_1 = x_1$, $\varphi_2 = x_2$, $\varphi_3 = x_4$, and hence

$$(\nabla\varphi(x), f(x)) = (\lambda_1 x_1, \lambda_2 x_2 - a x_2^2 x_4 + b x_3^2, -\omega x_3 - c x_3 x_4 + a x_2^3 + \lambda_4 x_4^3)^{\mathrm{T}}$$

Therefore, we obtain from formula (2.1)

$$V_a = [\lambda_1^2 x_1^2 + (\lambda_2 x_2 - a x_2^2 x_4 + b x_3^2)^2 + (-\omega x_3 - c x_3 x_4 + a x_2^3 + \lambda_4 x_4^3)^2]^m \times$$

$$\times [\lambda_1 x_1^2 + (\lambda_2 x_2 - a x_2^2 x_4 + b x_3^2) x_2 + (-\omega x_3 - c x_3 x_4 + a x_2^3 + \lambda_4 x_4^3) x_4] \qquad (2.4)$$

Representation (2.2) for system (1.5) has the form

$$f_M = (\lambda_1 x_1, \lambda_2 x_2 - a x_2^2 x_4, \omega x_4 + c x_4^2 - b x_2 x_3^2, -c x_3 x_4 + a x_2^3 + \lambda_4 x_4^3)^{\mathrm{T}}$$

$$f_N = (0, b x_3^2, 0, -\omega x_3)^{\mathrm{T}} \qquad (2.5)$$

Taking formulae (2.3) and (2.5) into account, we obtain the following simplified expression for the function V_a

$$V_a = (b^2 x_3^4 + \omega^2 x_3^2)^m (b x_2 x_3^2 - \omega x_3 x_4) \qquad (2.6)$$

The derivatives of the functions (2.4) and (2.6) in the set M take the value

$$\dot V_a = (b^2 x_3^4 + \omega^2 x_3^2)^{m+1}$$

3. Construction of the Lyapunov function

Suppose asymptotic stability of the zero solution is established for system (1.1) using the Barbashin–Krasovskii theorem. Then, by Massera's theorem,[8] a positive-definite function exists the derivative of which will be a negative-definite function. We will construct such a function using the function which occurs in the Barbashin–Krasovskii theorem and the additional functions introduced in Section 2.

Theorem 5. Suppose system (1.1) satisfies the Barbashin–Krasovskii theorem and the set $M = \{x : \dot V(x) = 0\}$ is defined by a single vector function: $\varphi(x) = 0$; $\nabla\varphi(x) \neq 0$ with $x \in M, x \neq 0$. We will assume that $f(x)$, $\varphi(x)$ and $V(x)$ are functions that are differentiable a sufficient number of times; the sign-definiteness of the function $V(x)$ is determined by a form of order β_s; the sign-positiveness of $\dot V(x)$ and the inequality $(\nabla\varphi(x), f(x)) \neq 0$ are defined by terms of the expansion in the neighbourhood of zero of finite order. Then numbers m and α exist such that the function $V_f(x) = V_s(x) + \alpha V_a(x)$ is positive-definite and its derivative $\dot V_f(x)$ is negative-definite; here $V_s(x) = V(x)$, and the function $V_a(x)$ is defined by formula (2.3).

Proof. By virtue of the above assumptions for establishing the sign-definiteness of the functions $V_f(x)$, $\dot V_f(x)$, it is sufficient to analyse the expansions of $V_s(x)$, $\dot V_s(x)$ and $V_a(x)$, $\dot V_a(x)$. Since the positive-definiteness of the function $V_s(x)$ is determined by a form of order β_s, while the expansion of the function $f_N(x)$ can begin with a first-order form, taking $m \geq \beta_s/2$ and any α, we obtain that the function $V_f(x)$ will be positive-definite, since the additional terms of the function $V_a(x)$ will have an order higher than β_s, and do not change the positive-definiteness of the function $V_s(x)$.

To analyse the expansions of $\dot V_s$, $\dot V_a$ we will convert system (1.1), introducing the variables

$$y = (y_1, \ldots, y_k) = \varphi(x), z = (z_1, \ldots, z_{n-k}) = (x_{k+1}, \ldots, x_n)$$

renumbering, if necessary, the variables x_i so that this transformation is non-degenerate. Note that the set M is defined by the equation $y = 0$. We will write system (1.1) in the form (2.2)

$$\dot y = f_1(y, z) = f_{1M}(y, z) + f_{1N}(z), \quad \dot z = f_2(y, z) = f_{2M}(y, z) + f_{2N}(z) \qquad (3.1)$$

According to the definition of the functions f_M and f_N we have

$$f_M^{\mathrm{T}} = (f_{1M}^{\mathrm{T}}(y, z), f_{2M}^{\mathrm{T}}(y, z)), \quad f_N^{\mathrm{T}} = (f_{1N}^{\mathrm{T}}(z), f_{2N}^{\mathrm{T}}(z))$$

and for function (2.3) we obtain the expression

$$V_a = f_{1N}^{2m}(z)(f_{1N}(z), y)$$

where

$$f_{1N}(z) = (\nabla\varphi, f_N) \neq 0 \quad \text{for } z \neq 0$$

We will use the expression obtained to calculate the derivative of the function V_a according to system (3.1). We have

$$\dot V_a = f_{1N}^{2m+2}(z) + f_{1N}^{2m}(z)(f_{1N}(z), f_{1M}(y, z)) + f_{1N}^{2m}(z)((\nabla f_{1N}(z), \;\; f_{2M}(y, z) + f_{2N}(z)), y) +$$

$$+ 2m f_{1N}^{2m-2}(z)((\nabla f_{1N}(z), f_{2M}(y, z) + f_{2N}(z)), \;\; f_{1N}(z))(f_{1N}(z), y) \qquad (3.2)$$

We will represent the derivative $\dot{V}_f$ in the form

$$\dot{V}_f = \dot{V}_s(y) + \alpha f_{1N}^{2m+2}(z) + \dot{V}_{fa}(y,z) \tag{3.3}$$

In view of the condition $f_{1N}(z) \neq 0$ for $z \neq 0$ we conclude that for $\alpha < 0$ the function $\dot{V}_s(y) + \alpha f_{1N}^{2m+2}(z)$ is negative-definite. To establish the negative-definiteness of $\dot{V}_f$ we will analyse the effect of $\dot{V}_{fa}(y,z)$. We will estimate the smallness of the terms $\dot{V}_f$, using representation (3.3). Suppose β_1 is the maximum order of the form in the expansion of $\dot{V}_s$, which determines its sign-constancy. We obtain from formulae (3.2) and (3.3)

$$\dot{V}_{fa}(y,z) = O(\|f_{1N}(z)\|^{2m})O(\|y\|)$$

Here we have taken into account the fact that the expansion of $f_{1M}(y,z)$ with respect to y begins with terms of power no lower than the first. Since the expansion of $f_{1N}(z)$ begins with terms no lower than the first power of z, $||f_{1N}(z)|| \sim o(||z||)$ and $\dot{V}_{fa}(y,z) = o(||z||^{2m})O(||y||)$. We will estimate the effect of the function $\dot{V}_{fa}(y,z)$ for $y \sim z^k$ when $k > 0$. For $0 < k \leq 2$, assuming $m > \beta_1$, we have $\dot{V}_{fa} \sim o(\dot{V}_s)$, while for $k > 2$ we have $\dot{V}_{fa}(y,z) = o(||z||^{2m+2})$. Hence it follows that the function $\dot{V}_{fa}(y,z)$, with the above assumptions, does not disturb the negative-definiteness of $\dot{V}_s(y) + \alpha f_{1N}^{2m+2}(z)$. Summing up the discussions, we can assert that the theorem holds if we choose

$$m = \max([\beta_s/2] + 1, \beta_1), \quad \alpha < 0$$

We will use Theorem 5 to construct a Lyapunov function with sign-definite derivative in Examples 1 and 2.

Example 1. In this case

$$V_s = x^2 + y^2, \quad \dot{V}_s = -2x^4$$

hence $\beta_s = 2$ and $\beta_1 = 4$. We take $m = 4$ and $\alpha = -1$ and we obtain as the Lyapunov function

$$V_f = x^2 + y^2 - c^7 x y^{21}$$

We find

$$\dot{V}_f = -2x^4 - c^8 y^{24} - c^7 y^{20} x[-21ax^3 - (21b+1)x^2 y - (21c-a)xy^2 + by^3]$$

Example 2. In this case

$$V_s = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad \dot{V}_s = 2\lambda_1 x_1^2 + 2\lambda_2 x_2^2 + 2\lambda_4 x_4^4$$

and hence $\beta_s = 2$ and $\beta_1 = 4$. We take $m = 4$ and $\alpha = -1$ and we obtain as the Lyapunov function

$$V_f = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_3 X_1^4 X_2; \quad X_1 = b^2 x_3^4 + \omega^2 x_3^2, \quad X_2 = bx_2 x_3 - \omega x_4$$

We find

$$\dot{V}_f = 2\lambda_1 x_1^2 + 2\lambda_2 x_2^2 + 2\lambda_4 x_4^4 - X_1^5 - [(8x_3^2(2b^2 x_3^2 + \omega^2)X_2 +$$
$$+ (2bx_2 x_3 - \omega x_4))(\omega x_4 + cx_4^2 - bx_2 x_3)X_1 +$$
$$+ bx_2 x_3^2(\lambda_2 - ax_2 x_4)X_1 - \omega x_3(-cx_3 x_4 + ax_2^3 + \lambda_4 x_4^3)X_1]X_1^3$$

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